Relativistic Quantum Mechanics

F755, academic year 2011 — **Prof. M. Kastner**

UNIVERSITEIT STELLENBOSCH UNIVERSITY

Problem sheet 4

Submission deadline: 22 August 2011

Problem 8: Symmetries of the free Dirac equation

3.5 points

Compute the commutators of the free Dirac-Hamiltonian $H = \boldsymbol{\alpha}\cdot\boldsymbol{P}+\beta m_0$ with the following observables:

(a) momentum $\boldsymbol{P} = -\mathrm{i}\boldsymbol{\nabla}$,

(b) orbital angular momentum $\boldsymbol{L} = \boldsymbol{X} \times \boldsymbol{P}$,

(c) spin $\boldsymbol{S} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}$,

(d) total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$,

(e) $\boldsymbol{S}\cdot\boldsymbol{P}$.

Problem 9: Zitterbewegung

(written) 5 points

Consider a system whose time evolution is governed by the free Dirac-Hamiltonian

$$H = \boldsymbol{\alpha}\cdot\boldsymbol{P} + \beta m_0.$$

(a) Compute the time derivative $\dot{\boldsymbol{X}}(t) = \mathrm{i}[H, \boldsymbol{X}(t)]$ of the position operator $\boldsymbol{X}(t)$ in the Heisenberg picture. Is this a physically reasonable velocity operator?

(b) Show

$$\ddot{\boldsymbol{X}}(t) = 2\mathrm{i}H\boldsymbol{F}(t) \qquad \text{where} \quad \boldsymbol{F}(t) \equiv \dot{\boldsymbol{X}}(t) - \boldsymbol{P}H^{-1}.$$

(c) Show that $\{\boldsymbol{F}(0), H\} = 0$ and that the vanishing of this anti-commutator implies $\boldsymbol{F}(t) = \mathrm{e}^{2\mathrm{i}Ht}\boldsymbol{F}(0)$. Now integrate the operator $\dot{\boldsymbol{X}}$ from (a) with respect to time.
(Result: $\boldsymbol{X}(t) = \boldsymbol{X}(0) + \boldsymbol{P}H^{-1}t - \frac{\mathrm{i}}{2}H^{-1}(\mathrm{e}^{2\mathrm{i}Ht} - 1)\boldsymbol{F}(0)$.

(d) The oscillations occurring in the result of (c) are termed Zitterbewegung. Give an estimate for the amplitude and frequency of these oscillations.

(e) What is the expectation value $\langle E|\boldsymbol{F}|E\rangle$ of $\boldsymbol{F}$ with respect to an eigenstate $|E\rangle$ of H?

Problem 10: Massless Dirac particle

5 points

Let $H = \boldsymbol{\alpha}\cdot\boldsymbol{P}$ be the Dirac-Hamiltonian of a free particle of rest mass $m_0 = 0$.

(a) Determine the eigenvalues of $\gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, without making use of any representation of the γ-matrices.

(b) Show that $\boldsymbol{P}$, H and γ^5 are pairwise commuting operators.

(c) Verify explicitly that

$$\Psi_{\boldsymbol{p},\pm}(t,\boldsymbol{r}) = \begin{pmatrix}\mathbb{1}_2\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E(p)}\end{pmatrix} h_\pm(\boldsymbol{p})\mathrm{e}^{\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{r}-E(p)t)},$$

is a simultaneous eigenfunction of $\boldsymbol{P}$, H and γ^5 with eigenvalues $\boldsymbol{p}$, $E(p) = \pm|\boldsymbol{p}|$ and $\pm$.
Reminder: $h_\pm(\boldsymbol{p})$ was defined in one of the lectures as a two-component function satisfying

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})h_\pm(\boldsymbol{p}) = \pm p\, h_\pm(\boldsymbol{p}).$$